**Definition.** *A number $l$ is said to be the limit of function $f(x, y)$ as $(x, y) \to (a, b)$, if for any real number $\varepsilon > 0$, there exists a real number $\delta > 0$ such that for all points $(x, y)$ other than $(a, b)$*

$$|f(x, y) - l| < \varepsilon \text{ for } 0 < |x - a| < \delta \text{ and } 0 < |y - b| < \delta$$

[Based on square neighbourhood of a point]

or

$$|f(x, y) - l| < \varepsilon \text{ for } 0 < \sqrt{(x-a)^2 + (y-b)^2} < \delta$$

[Based on circular neighbourhood of a point]

and we write $\displaystyle\lim_{\substack{x \to a \\ y \to b}} f(x, y) = l.$

**Example 4.** ***Let $f : R^2 \to R$ be defined as***

$$f(x, y) = \begin{cases} \dfrac{xy}{x^2 + y^2}, & (x, y) \neq (0, 0) \\ 0, & (x, y) = (0, 0) \end{cases}$$

[M.D.U. 2014, 11]

***Show that*** $\lim\limits_{(x, y) \to (0, 0)} f(x, y)$ ***does not exist.***

**Solution.** If $(x, y) \to (0, 0)$ along the line $y = mx$, but $(x, y) \neq (0, 0)$, then

$$f(x, y) = \frac{xy}{x^2 + y^2} = \frac{x(mx)}{x^2 + m^2x^2} = \frac{m}{1+m^2}$$

$$\therefore \quad \lim_{\substack{(x, y) \to (0, 0) \\ \text{along } y = mx}} f(x, y) = \frac{m}{1+m^2}$$

This limit is different for different values of $m$. Since the limit of the function depends on the manner of approach to $(0, 0)$, therefore, $\lim_{(x, y) \to (0, 0)} f(x, y)$ does not exist.

**Example 5.** *Let $f: R^2 \to R$ be defined as $f(x, y) = \dfrac{x - y}{x + y}$, $x \neq 0$, $y \neq 0$.*

*Show that $\lim_{(x, y) \to (0, 0)} f(x, y)$ does not exist.* [K.U. 2014, 10; M.D.U. 20

**Solution.** Let $(x, y) \to (0, 0)$ along the path $y = mx$, where $m$ is any real number.

As $x \to 0$, then $y \to 0$ along the path $y = mx$.

Now,
$$\lim_{(x, y) \to (0, 0)} f(x, y) = \lim_{(x, y) \to (0, 0)} \frac{x - y}{x + y}$$

$$= \lim_{x \to 0} \frac{x - mx}{x + mx} = \lim_{x \to 0} \left(\frac{1-m}{1+m}\right),$$

which is different for different values of $m$.

$$\therefore \quad \lim_{(x, y) \to (0, 0)} f(x, y) \text{ does not exist.}$$

## 4.9. CONTINUITY OF A FUNCTION OF TWO VARIABLES

**Definition 1.** *A function $f(x, y)$ is said to be continuous at a point $(a, b)$ if $\lim_{(x, y) \to (a, b)} f(x, y)$ exists and is equal to $f(a, b)$.*

**Definition 2.** *A function $f(x, y)$ is continuous at a point $(a, b)$ if, given any $\varepsilon > 0$, we can find a real number $\delta > 0$ such that*

$$|f(x, y) - f(a, b)| < \varepsilon$$

*for all $(x, y)$ that satisfy $|(x, y) - (a, b)| < \delta$.*

**Remark.**

Similarly we can define continuity of a function of three variables.

A function $f(x, y, z)$ is said to be continuous at a point $(a, b, c)$ if :

$$\lim_{(x, y, z) \to (a, b, c)} f(x, y, z) \text{ exists and is equal to } f(a, b, c).$$

or if given $\varepsilon > 0$, however small, there exists a real number $\delta$, such that

$$|f(x, y, z) - f(a, b, c)| < \varepsilon \text{ for } |(x, y, z) - (a, b, c)| < \delta.$$

## 4.10. ALGEBRA OF CONTINUOUS FUNCTIONS

If $f(x, y)$ and $g(x, y)$ are continuous functions of $x$ and $y$ on a set D and $k$ is any constant, then

(i) $f(x, y) + g(x, y)$ is continuous on D

(ii) $f(x, y) - g(x, y)$ is continuous on D

(iii) $f(x, y) \cdot g(x, y)$ is continuous on D

(iv) $k f(x, y)$ is continuous on D

(v) $\dfrac{f(x, y)}{g(x, y)}$ is continuous on D except for those points $(x_0, y_0)$ where $g(x_0, y_0) = 0$.

**Example 6.** *Prove that the function $f : A \to R, A \subset R^2$ defined by*

$$f(x, y) = \begin{cases} x \sin \dfrac{1}{y}, & y \neq 0 \\ 0, & y = 0 \end{cases}$$

[K.U. 2016, 05; C.D.L.U. 2013]

*is continuous at $(0, 0)$.*

**Solution.** Let $\varepsilon > 0$ be any given number

Here $$|f(x, y) - f(0, 0)| = \left| x \sin \frac{1}{y} - 0 \right|$$

$$= \left| x \sin \frac{1}{y} \right|$$

$$= |x| \left| \sin \frac{1}{y} \right|$$

$$\leq |x| \qquad \left[\because \left|\sin\frac{1}{y}\right| \leq 1\right]$$

$$\therefore \qquad |f(x, y) - f(0, 0)| \leq |x - 0| \qquad ...(1)$$

Let $|x - 0| < \varepsilon$ and $|y - 0| < \varepsilon$

From (1), we have $|f(x, y) - f(0, 0)| < \varepsilon$

Hence, $f(x, y)$ is continuous at $(0, 0)$.

**Example 7.** ***Prove that the function $f : A \to R$, $A \subset R^2$ defined by***

$$f(x, y) = \begin{cases} 1 & \text{if } y \neq 0 \\ 0 & \text{if } y = 0 \end{cases}$$

***is not continuous at $(a, b)$ if $b = 0$.***

**Solution.** We have $f(a, 0) = 0$ for all $a$.

$$\therefore \qquad \lim_{(x, y) \to (a, 0)} f(x, y) = \lim_{\substack{x \to a \\ y \to 0}} f(x, y)$$

$$= \lim_{\substack{x \to a \\ y \to 0}} 1 \qquad [\because \textit{When } y \to 0, y \neq 0,\ f(x, y) = 1]$$

$$= 1$$

But $\qquad f(a, 0) = 0$

Thus $\qquad \lim_{(x, y) \to (a, 0)} f(x, y) \neq f(a, 0)$

Hence, $f$ is discontinuous at $(a, 0)$ for all $a$.

**Example 8.** ***Show that the function $f : R^2 \to R$ defined by***

$$f(x, y) = \begin{cases} xy\,\dfrac{(x^2 - y^2)}{x^2 + y^2}; & (x, y) \neq (0, 0) \\ 0; & \text{otherwise} \end{cases}$$

***is continuous at $(0, 0)$.*** [K.U. 2015, 06; M.D.U. 2015, 13, 04, 02]

**Solution.** Let $\varepsilon > 0$ be a given real number.

Here $\qquad |f(x, y) - f(0, 0)| = \left|\dfrac{xy(x^2 - y^2)}{x^2 + y^2} - 0\right|$

$$= |xy| \left|\frac{x^2 - y^2}{x^2 + y^2}\right|$$

$$\le |xy| \qquad \left[\begin{array}{l} \text{Since } |x^2 - y^2| \le |x^2 + y^2| \\ \therefore \quad \left|\dfrac{x^2 - y^2}{x^2 + y^2}\right| \le 1 \end{array}\right]$$

$$|f(x, y) - f(0, 0)| \le |x|\,|y|$$

$$< \varepsilon \text{ if } |x - 0| < \sqrt{\varepsilon} \quad \text{and} \quad |y - 0| < \sqrt{\varepsilon}$$

$$\therefore \quad \lim_{(x, y) \to (0, 0)} f(x, y) = f(0, 0)$$

Hence, $f(x, y)$ is continuous at $(0, 0)$.

**Example 9.** ***Show that the function***

$$f(x, y) = \begin{cases} \dfrac{xy}{\sqrt{x^2 + y^2}} ; & (x, y) \ne (0, 0) \\ 0 ; & (x, y) = (0, 0) \end{cases}$$

*[K.U. 2017, 15, 06; M.D.U. 2016, 15, 13, 04, 02]*

***continuous at (0, 0).***

**Solution.** Let $\varepsilon > 0$ be a given real number.

Here $$\left|\frac{xy}{\sqrt{x^2 + y^2}} - 0\right| = \frac{|xy|}{\sqrt{x^2 + y^2}}$$

$$= r\,|\cos\theta \sin\theta| \text{ for } x = r\cos\theta, y = r\sin\theta$$

$$\le r$$

$$= \sqrt{x^2 + y^2}$$

$$< \varepsilon, \text{ if } x^2 < \frac{1}{2}\varepsilon^2, \; y^2 < \frac{1}{2}\varepsilon^2$$

$$\text{i.e., if } |x| < \frac{\varepsilon}{\sqrt{2}}, \; |y| < \frac{\varepsilon}{\sqrt{2}}$$

$$\therefore \quad \left|\frac{xy}{\sqrt{x^2 + y^2}} - 0\right| < \varepsilon \text{ if } |x - 0| < \frac{\varepsilon}{\sqrt{2}}, |y - 0| < \frac{\varepsilon}{\sqrt{2}}$$

$\therefore$ $\lim\limits_{(x, y) \to (0, 0)} f(x, y) = f(0, 0)$ and so $f(x, y)$ is continuous at $(0, 0)$.

## 5.2. RULES OF PARTIAL DIFFERENTIATION

1. If $u$ is a function of $x, y$ and we have to differentiate partially w.r.t. $x$, then $y$ is *as constant*. Similarly to differentiate $u$ partially w.r.t. $y$, *$x$ is treated as constant*.

2. If $z = u \pm v$, where $u$, $v$ are functions of $x$ and $y$, then

$$\frac{\partial z}{\partial x} = \frac{\partial u}{\partial x} \pm \frac{\partial v}{\partial x} \quad \text{and} \quad \frac{\partial z}{\partial y} = \frac{\partial u}{\partial y} \pm \frac{\partial v}{\partial y}.$$

3. If $z = uv$, where $u$, $v$ are functions of $x$ and $y$, then

$$\frac{\partial z}{\partial x} = \frac{\partial}{\partial x}(uv) = u\,\frac{\partial v}{\partial x} + v\,\frac{\partial u}{\partial x}$$

and

$$\frac{\partial z}{\partial y} = \frac{\partial}{\partial y}(uv) = u\,\frac{\partial v}{\partial y} + v\,\frac{\partial u}{\partial y}.$$

4. If $z = \dfrac{u}{v}$, where $u$, $v$ are functions of $x$ and $y$, then

$$\frac{\partial z}{\partial x} = \frac{\partial}{\partial x}\left(\frac{u}{v}\right) = \frac{v\,\dfrac{\partial u}{\partial x} - u\,\dfrac{\partial v}{\partial x}}{v^2}$$

and

$$\frac{\partial z}{\partial y} = \frac{\partial}{\partial y}\left(\frac{u}{v}\right) = \frac{v\,\dfrac{\partial u}{\partial y} - v\,\dfrac{\partial v}{\partial y}}{v^2}$$

## SOLVED EXAMPLES

**Example 1.** ***If*** $z = \dfrac{1}{\sqrt{x^2+y^2}}$; ***find*** $\dfrac{\partial z}{\partial x}$ ***and*** $\dfrac{\partial z}{\partial y}$. [M.D.U. 2017, 15; K.U. 2014]

**Solution.** Here $z = (x^2+y^2)^{-1/2}$

$$\therefore \quad \frac{\partial z}{\partial x} = -\frac{1}{2}(x^2+y^2)^{-3/2} \cdot 2x = -\frac{x}{(x^2+y^2)^{3/2}}$$

and
$$\frac{\partial z}{\partial y} = -\frac{1}{2}(x^2+y^2)^{-3/2} \cdot 2y = -\frac{y}{(x^2+y^2)^{3/2}}.$$

**Example 2.** ***If*** $z = \tan^{-1}\left(\dfrac{x}{y}\right)$, ***find*** $\dfrac{\partial z}{\partial x}$ ***and*** $\dfrac{\partial z}{\partial y}$. ***Also verify that*** $\dfrac{\partial^2 z}{\partial x \partial y} = \dfrac{\partial^2 z}{\partial y \partial x}$. [M.D.U. 2016]

**Solution.** Here $z = \tan^{-1}\left(\dfrac{x}{y}\right)$

$$\therefore \quad \frac{\partial z}{\partial x} = \frac{1}{1+\left(\frac{x}{y}\right)^2} \cdot \frac{1}{y} = \frac{y}{y^2+x^2} \quad ...(1)$$

and
$$\frac{\partial z}{\partial y} = \frac{1}{1+\left(\frac{x}{y}\right)^2} \cdot \frac{-x}{y^2} = -\frac{x}{y^2+x^2} \quad ...(2)$$

Differentiating (1) partially w.r.t. $y$, we have

$$\frac{\partial^2 z}{\partial y \partial x} = \frac{(x^2+y^2)\cdot 1 - y \cdot 2y}{(x^2+y^2)^2} = \frac{x^2-y^2}{(x^2+y^2)^2} \quad ...(3)$$

Differentiating (2) partially w.r.t. $x$, we have

$$\frac{\partial^2 z}{\partial x \partial y} = -\frac{(x^2+y^2)\cdot 1 - x\cdot(2x)}{(x^2+y^2)^2} = \frac{x^2-y^2}{(x^2+y^2)^2} \quad ...(4)$$

From (3) and (4), we have $\dfrac{\partial^2 z}{\partial y \partial x} = \dfrac{\partial^2 z}{\partial x \partial y}$.

**Example 3.** ***If*** $u = \log(x^2+y^2)$, ***then prove that (i)*** $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = 0$ ***(ii)*** $\dfrac{\partial^2 u}{\partial x \partial y} = \dfrac{\partial^2 u}{\partial y \partial x}$

**Solution.** Here $u = \log(x^2+y^2)$



$$\therefore \quad \frac{\partial u}{\partial x} = \frac{1}{x^2+y^2} \cdot 2x = \frac{2x}{x^2+y^2} \quad ...(1)$$

and
$$\frac{\partial u}{\partial y} = \frac{1}{x^2+y^2} \cdot 2y = \frac{2y}{x^2+y^2} \quad ...(2)$$

(*i*) Differentiating (1) partially w.r.t. $x$, we have

$$\frac{\partial^2 u}{\partial x^2} = \frac{(x^2+y^2).2 - 2x(2x)}{(x^2+y^2)^2} = \frac{2y^2-2x^2}{(x^2+y^2)^2} \quad ...(3)$$

Differentiating (2) partially w.r.t. $y$, we have

$$\frac{\partial^2 u}{\partial y^2} = \frac{(x^2+y^2).2 - 2y(2y)}{(x^2+y^2)^2} = \frac{2x^2-2y^2}{(x^2+y^2)^2} \quad ...(4)$$

Adding (3) and (4), we get

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{2y^2-2x^2}{(x^2+y^2)^2} + \frac{2x^2-2y^2}{(x^2+y^2)^2} = 0$$

(*ii*) Differentiating (1) partially w.r.t. $y$, we have

$$\frac{\partial^2 u}{\partial y \partial x} = \frac{(x^2+y^2).0 - 2x(2y)}{(x^2+y^2)^2} = \frac{-4xy}{(x^2+y^2)^2} \quad ...(5)$$

Differentiating (2) partially w.r.t. $x$, we have

$$\frac{\partial^2 u}{\partial x \partial y} = \frac{(x^2+y^2).0 - 2y(2x)}{(x^2+y^2)^2} = \frac{-4xy}{(x^2+y^2)^2} \quad ...(6)$$

$\therefore$ From (5) and (6), $\dfrac{\partial^2 u}{\partial x \partial y} = \dfrac{\partial^2 u}{\partial y \partial x}$.

**Example 4.** ***If*** $u = \log \dfrac{x^2+y^2}{xy}$, ***then find*** $\dfrac{\partial^2 u}{\partial y \partial x}$ ***and*** $\dfrac{\partial^2 u}{\partial x \partial y}$ ***and hence show that*** $\dfrac{\partial^2 u}{\partial y \partial x} = \dfrac{\partial^2 u}{\partial x \partial y}$. [M.D.U. 2015]

**Solution.** Here $u = \log(x^2+y^2) - \log x - \log y$

$$\therefore \quad \frac{\partial u}{\partial x} = \frac{1}{x^2+y^2} \cdot 2x - \frac{1}{x} \quad ...(1)$$

and
$$\frac{\partial u}{\partial y} = \frac{1}{x^2+y^2} \cdot 2y - \frac{1}{y} \quad ...(2)$$

$$\therefore \quad \frac{\partial^2 u}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x}\right) = \frac{\partial}{\partial y}\left(\frac{2x}{x^2+y^2} - \frac{1}{x}\right) \quad \text{[Using (1)]}$$

$$= 2x(-1)(x^2+y^2)^{-2}(2y) - 0 = \frac{-4xy}{(x^2+y^2)^2}$$

Also,
$$\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial y}\right) = \frac{\partial}{\partial x}\left(\frac{2y}{x^2+y^2} - \frac{1}{y}\right) \quad \text{[Using (2)]}$$

$$= 2y(-1)(x^2+y^2)^{-2}\cdot(2x) - 0$$

$$= \frac{-4xy}{(x^2+y^2)^2}$$

Hence, $\dfrac{\partial^2 u}{\partial y \partial x} = \dfrac{\partial^2 u}{\partial x \partial y}$.

**Note.** In almost all case we come across, if $u$ is a continuous function of $x$ and $y$, then $\dfrac{\partial^2 u}{\partial x \partial y} = \dfrac{\partial^2 u}{\partial y \partial x}$.

**Example 5.** *If $u = x^y + y^x$, prove that* $\dfrac{\partial^2 u}{\partial x \partial y} = \dfrac{\partial^2 u}{\partial y \partial x}$. [K.U. 2015]

**Solution.** Here $u = x^y + y^x$

$$\therefore \quad \frac{\partial u}{\partial x} = yx^{y-1} + y^x \log y \quad ...(1)$$

and
$$\frac{\partial u}{\partial y} = x^y \log x + xy^{x-1} \quad ...(2)$$

$$\therefore \quad \frac{\partial^2 u}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x}\right) = \frac{\partial}{\partial y}(yx^{y-1} + y^x \log y)$$

$$= 1 \,.\, x^{y-1} + y \,.\, x^{y-1} \,.\, \log x + xy^{x-1}\log y + y^x \,.\, \frac{1}{y}$$

$$= x^{y-1}(1 + y\log x) + y^{x-1}(1 + x\log y) \quad ...(3)$$

Also,
$$\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial y}\right) = \frac{\partial}{\partial x}(x^y \log x + xy^{x-1})$$

$$= yx^{y-1}\log x + x^y\left(\frac{1}{x}\right) + 1.y^{x-1} + xy^{x-1}\log y$$



$$= x^{y-1}(y\log x + 1) + y^{x-1}(1 + x\log y) \quad ...(4)$$

$\therefore$ From (3) and (4), $\dfrac{\partial^2 u}{\partial x \partial y} = \dfrac{\partial^2 u}{\partial y \partial x}$.

**Example 6.** *If $x = r\cos\theta$ and $y = r\sin\theta$, prove that*

$$\frac{\partial^2 r}{\partial x^2}\cdot\frac{\partial^2 r}{\partial y^2} = \left(\frac{\partial^2 r}{\partial x \partial y}\right)^2.$$

[M.D.U. 2016, 15, 10; K.U. 2014]

**Solution.** The given equations are

$$x = r\cos\theta$$
$$y = r\sin\theta$$

To eliminate $\theta$ from the given equations, by squaring and adding these, we have

$$r^2 = x^2 + y^2$$
$$r = (x^2+y^2)^{1/2}$$

*i.e.,*
$$\frac{\partial r}{\partial x} = \frac{1}{2}(x^2+y^2)^{-1/2}\cdot 2x = \frac{x}{(x^2+y^2)^{1/2}} = \frac{x}{r}$$

$\therefore$
$$\frac{\partial r}{\partial y} = \frac{1}{2}(x^2+y^2)^{-1/2}\cdot 2y = \frac{y}{r}$$

and

Again,
$$\frac{\partial^2 r}{\partial x^2} = \frac{r\cdot 1 - x\dfrac{\partial r}{\partial x}}{r^2} = \frac{r - x\cdot\dfrac{x}{r}}{r^2}$$

$$= \frac{r^2 - x^2}{r^3} = \frac{y^2}{r^3} \quad ...(1) \quad [\because r^2 = x^2 + y^2]$$

Similarly,
$$\frac{\partial^2 r}{\partial y^2} = \frac{x^2}{r^3} \quad ...(2)$$

$$\therefore \quad \text{L.H.S.} = \frac{\partial^2 r}{\partial x^2}\cdot\frac{\partial^2 r}{\partial y^2} = \frac{y^2}{r^3}\cdot\frac{x^2}{r^3} = \frac{x^2y^2}{r^6}$$

$$\text{R.H.S.} = \left(\frac{\partial^2 r}{\partial x \partial y}\right)^2 = \left[\frac{\partial}{\partial x}\cdot\frac{\partial r}{\partial y}\right]^2 = \left[\frac{\partial}{\partial x}\left(\frac{y}{r}\right)\right]^2 = \left(-\frac{y}{r^2}\cdot\frac{\partial r}{\partial x}\right)^2$$

$$= \left(-\frac{y}{r^2}\cdot\frac{x}{r}\right)^2 = \left(\frac{-xy}{r^3}\right)^2 = \frac{x^2y^2}{r^6}.$$

$\therefore$ L.H.S. = R.H.S. Hence we get the desired result.

**Example 7.** *If $u = f(r)$ where $r = \sqrt{x^2+y^2}$, prove that*

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = f''(r) + \frac{1}{r} f'(r).$$ [K.U. 2007]

**Solution.** Here $r = \sqrt{x^2+y^2}$

$$\therefore \quad \frac{\partial r}{\partial x} = \frac{x}{r};\ \frac{\partial r}{\partial y} = \frac{y}{r};\ \frac{\partial^2 r}{\partial x^2} = \frac{y^2}{r^3};\ \frac{\partial^2 r}{\partial y^2} = \frac{x^2}{r^3}$$ [Ref. Example 6]

Now, $$\frac{\partial u}{\partial x} = f'(r)\cdot\frac{\partial r}{\partial x}$$

and $$\frac{\partial^2 u}{\partial x^2} = f'(r)\cdot\frac{\partial^2 r}{\partial x^2} + \frac{\partial r}{\partial x}\cdot f''(r)\cdot\frac{\partial r}{\partial x}$$

i.e., $$\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 r}{\partial x^2} f'(r) + \left(\frac{\partial r}{\partial x}\right)^2 f''(r) \qquad ...(1)$$

Similarly, $$\frac{\partial^2 u}{\partial y^2} = \frac{\partial^2 r}{\partial y^2} f'(r) + \left(\frac{\partial r}{\partial y}\right)^2 f''(r) \qquad ...(2)$$

Adding (1) and (2), we have

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \left(\frac{\partial^2 r}{\partial x^2} + \frac{\partial^2 r}{\partial y^2}\right) f'(r) + \left\{\left(\frac{\partial r}{\partial x}\right)^2 + \left(\frac{\partial r}{\partial y}\right)^2\right\} f''(r)$$

$$= \left(\frac{y^2}{r^3} + \frac{x^2}{r^3}\right) f'(r) + \left(\frac{x^2}{r^2} + \frac{y^2}{r^2}\right) f''(r)$$

$$= \frac{1}{r} f'(r) + f''(r) \qquad [\because x^2+y^2=r^2]$$

**Example 8.** *If $u = \log(x^3+y^3+z^3-3xyz)$; show that*

$$\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = \frac{3}{x+y+z}.$$

**Solution.** Here $u = \log(x^3+y^3+z^3-3xyz)$

$$= \log(x+y+z)(x^2+y^2+z^2-yz-zx-xy)$$

$$= \log(x+y+z) + \log(x^2+y^2+z^2-yz-zx-xy)$$

$$\therefore \quad \frac{\partial u}{\partial x} = \frac{1}{x+y+z} + \frac{1}{(x^2+y^2+z^2-yz-zx-xy)}\cdot(2x-z-y)$$



$$= \frac{1}{x+y+z} + \frac{2x-y-z}{x^2+y^2+z^2-yz-zx-xy} \qquad ...(1)$$

Similarly, $$\frac{\partial u}{\partial y} = \frac{1}{x+y+z} + \frac{2y-z-x}{x^2+y^2+z^2-yz-zx-xy} \qquad ...(2)$$

and $$\frac{\partial u}{\partial z} = \frac{1}{x+y+z} + \frac{2z-x-y}{x^2+y^2+z^2-yz-zx-xy} \qquad ...(3)$$

Adding (1), (2) and (3), we get

$$\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = \frac{3}{x+y+z}.$$

**Example 9.** *If $z = x\cos\left(\frac{y}{x}\right) + \tan\left(\frac{y}{x}\right)$; show that*

$$x^2\frac{\partial^2 z}{\partial x^2} + 2xy\frac{\partial^2 z}{\partial x\partial y} + y^2\frac{\partial^2 z}{\partial y^2} = 0,\ (x \neq 0).$$

**Solution.** Here $z = x\cos\left(\frac{y}{x}\right) + \tan\left(\frac{y}{x}\right)$

$$\therefore \quad \frac{\partial z}{\partial x} = 1.\cos\left(\frac{y}{x}\right) + x.\left[-\sin\left(\frac{y}{x}\right)\cdot\left(\frac{-y}{x^2}\right)\right] + \sec^2\left(\frac{y}{x}\right)\left(\frac{-y}{x^2}\right)$$

$$= \cos\left(\frac{y}{x}\right) + \frac{y}{x}\sin\left(\frac{y}{x}\right) - \frac{y}{x^2}\sec^2\left(\frac{y}{x}\right) \qquad ...(1)$$

and $$\frac{\partial z}{\partial y} = x\left[-\sin\left(\frac{y}{x}\right)\cdot\left(\frac{1}{x}\right)\right] + \sec^2\left(\frac{y}{x}\right)\left(\frac{1}{x}\right)$$

$$= -\sin\left(\frac{y}{x}\right) + \frac{1}{x}\sec^2\left(\frac{y}{x}\right) \qquad ...(2)$$

$$\therefore \quad x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = x\cos\left(\frac{y}{x}\right) + y\sin\left(\frac{y}{x}\right) - \frac{y}{x}\sec^2\left(\frac{y}{x}\right) - y\sin\left(\frac{y}{x}\right) + \frac{y}{x}\sec^2\left(\frac{y}{x}\right)$$

or $$x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = x\cos\left(\frac{y}{x}\right) \qquad ...(3)$$

Differentiating (3) partially w.r.t. $x$, we have

$$1.\frac{\partial z}{\partial x}+x\frac{\partial^2 z}{\partial x^2}+y.\frac{\partial^2 z}{\partial x\,\partial y} = 1.\cos\left(\frac{y}{x}\right)+x\left[-\sin\left(\frac{y}{x}\right)\left(\frac{-y}{x^2}\right)\right]$$

or
$$\frac{\partial z}{\partial x}+x\frac{\partial^2 z}{\partial x^2}+y\frac{\partial^2 z}{\partial x\,\partial y} = \cos\left(\frac{y}{x}\right)+\frac{y}{x}\sin\left(\frac{y}{x}\right)$$

Differentiating (3) partially w.r.t. $y$, we have

$$x\frac{\partial^2 z}{\partial y\,\partial x}+1.\frac{\partial z}{\partial y}+y\frac{\partial^2 z}{\partial y^2} = x\left[-\sin\left(\frac{y}{x}\right)\left(\frac{1}{x}\right)\right]$$

or
$$x\frac{\partial^2 z}{\partial y\,\partial x}+\frac{\partial z}{\partial y}+y\frac{\partial^2 z}{\partial y^2} = -\sin\left(\frac{y}{x}\right)$$

Multiplying (4) by $x$ and (5) by $y$ and adding, we get

$$x\frac{\partial z}{\partial x}+x^2\frac{\partial^2 z}{\partial x^2}+xy\frac{\partial^2 z}{\partial x\,\partial y}+xy\frac{\partial^2 z}{\partial y\,\partial x}+y\frac{\partial z}{\partial y}+y^2\frac{\partial^2 z}{\partial y^2}$$
$$= x\cos\left(\frac{y}{x}\right)+y\sin\left(\frac{y}{x}\right)-y\sin\left(\frac{y}{x}\right)$$

or
$$x^2\frac{\partial^2 z}{\partial x^2}+2xy\frac{\partial^2 z}{\partial x\,\partial y}+y^2\frac{\partial^2 z}{\partial y^2} = x\cos\left(\frac{y}{x}\right)-\left(x\frac{\partial z}{\partial x}+y\frac{\partial z}{\partial y}\right)$$

or
$$x^2\frac{\partial^2 z}{\partial x^2}+2xy\frac{\partial^2 z}{\partial x\,\partial y}+y^2\frac{\partial^2 z}{\partial y^2} = x\cos\left(\frac{y}{x}\right)-x\cos\left(\frac{y}{x}\right)$$ [Using (3)]

or
$$x^2\frac{\partial^2 z}{\partial x^2}+2xy\frac{\partial^2 z}{\partial x\,\partial y}+y^2\frac{\partial^2 z}{\partial y^2} = 0.$$

**Example 10.** *If $z$ is a function of $x$ and $y$, given by the equation $x^x y^y z^z = c$, where $c$ is a constant, show that at $x = y = z$,* $\dfrac{\partial^2 z}{\partial x\,\partial y} = -[x\log ex]^{-1}$.

[K.U. 2015, 10, 05; M.D.U. 2015, 13]

**Solution.** Here $\quad x^x y^y z^z = c$

$$\therefore \quad z^z = \frac{c}{x^x y^y}$$

Taking log on both sides, we have

$$z\log z = \log c - x\log x - y\log y$$

Differentiating partially w.r.t. $x$, we have

$$z \cdot \frac{1}{z} \cdot \frac{\partial z}{\partial x} + (\log z) \frac{\partial z}{\partial x} = -[\log x + 1]$$

$$\frac{\partial z}{\partial x} = -\frac{1 + \log x}{1 + \log z} \qquad \text{...(1)}$$

Differentiating (1) partially w.r.t. $y$, we have

$$\frac{\partial^2 z}{\partial y \partial x} = -\frac{(-1)(1 + \log x)\frac{1}{z}\frac{\partial z}{\partial y}}{(1 + \log z)^2}$$

$$\frac{\partial^2 z}{\partial x \partial y} = \frac{1 + \log x}{z(1 + \log z)^2} \frac{\partial z}{\partial y} \qquad \text{...(2)}$$

Also, $$\frac{\partial z}{\partial y} = -\frac{1 + \log y}{1 + \log z}$$

Putting this value of $\frac{\partial z}{\partial y}$ in (2), we get

$$\frac{\partial^2 z}{\partial x \partial y} = -\frac{(1 + \log x)(1 + \log y)}{z(1 + \log z)^3}$$

$$\therefore \quad \left(\frac{\partial^2 z}{\partial x \partial y}\right)_{x=y=z} = -\frac{1}{x[1 + \log x]} \qquad \text{[Putting } y = z = x\text{]}$$

$$= -\frac{1}{x[\log e + \log x]} = -\frac{1}{x \log ex}$$

$$= -[x \log ex]^{-1}.$$

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## 5.4. HOMOGENEOUS FUNCTIONS

*A function $f(x, y)$ is said to be a homogeneous function of order n, if the degree of each of its terms in x and y is equal to n.*

In the expression, $a_0 x^n + a_1 x^{n-1} y + a_2 x^{n-2} y^2 + \dots + a_{n-1} x y^{n-1} + a_n y^n$, we observe that every term is of degree $n$. Thus it is an homogeneous function of degree $n$. It can also be written as $x^n \left\{ a_0 + a_1 \frac{y}{x} + a_2 \left(\frac{y}{x}\right)^2 + \dots + a_n \left(\frac{y}{x}\right)^n \right\}$.

## 5.6. EULER'S THEOREM ON HOMOGENEOUS FUNCTIONS

**If $u$ be a homogeneous function of degree $n$ in $x$ and $y$, then $x\dfrac{\partial u}{\partial x}+y\dfrac{\partial u}{\partial y}=nu$.**

[*M.D.U. 2014, 07, 04; K.U. 2004*]

**Proof.** Since $u$ is a homogeneous function of degree $n$ in $x$ and $y$, therefore we can express it as

$$u = x^n f\left(\frac{y}{x}\right)$$

## SOLVED EXAMPLES

**Example 1.** ***Verify Euler's theorem for the following:***

(i) $\dfrac{x^{1/4}+y^{1/4}}{x^{1/5}+y^{1/5}}$ (ii) $ax^2+2hxy+by^2$.

**Solution.** (i) Let $u = \dfrac{x^{1/4}+y^{1/4}}{x^{1/5}+y^{1/5}}$

$$\therefore \quad \frac{\partial u}{\partial x} = \frac{(x^{1/5}+y^{1/5})\,\dfrac{1}{4}\,x^{-3/4} - (x^{1/4}+y^{1/4})\,\dfrac{1}{5}\cdot x^{-4/5}}{(x^{1/5}+y^{1/5})^2}$$

and $$\frac{\partial u}{\partial y} = \frac{(x^{1/5}+y^{1/5})\frac{1}{4}\cdot y^{-3/4} - (x^{1/4}+y^{1/4})\frac{1}{5}\cdot y^{-4/5}}{(x^{1/5}+y^{1/5})^2}$$

$$\therefore \quad x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \frac{\frac{1}{4}x^{\frac{1}{4}}(x^{\frac{1}{5}}+y^{\frac{1}{5}}) - \frac{1}{5}x^{\frac{1}{5}}(x^{\frac{1}{4}}+y^{\frac{1}{4}}) + \frac{1}{4}\cdot y^{\frac{1}{4}}(x^{\frac{1}{5}}+y^{\frac{1}{5}}) - \frac{1}{5}y^{\frac{1}{5}}(x^{\frac{1}{4}}+y^{\frac{1}{4}})}{(x^{1/5}+y^{1/5})^2}$$

$$= \frac{\frac{1}{4}(x^{1/5}+y^{1/5})(x^{1/4}+y^{1/4}) - \frac{1}{5}(x^{1/4}+y^{1/4})(x^{1/5}+y^{1/5})}{(x^{1/5}+y^{1/5})^2}$$

$$= \frac{\frac{1}{20}(x^{1/5}+y^{1/5})(x^{1/4}+y^{1/4})}{(x^{1/5}+y^{1/5})^2}$$

$$\therefore \quad x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \frac{1}{20}\times\frac{x^{1/4}+y^{1/4}}{x^{1/5}+y^{1/5}} = \frac{1}{20}u \qquad \text{...(1)}$$

**Verification:**

Here $$u = \frac{x^{1/4}+y^{1/4}}{x^{1/5}+y^{1/5}}$$

$$= \frac{x^{1/4}\left[1+\left(\frac{y}{x}\right)^{1/4}\right]}{x^{1/5}\left[1+\left(\frac{y}{x}\right)^{1/5}\right]} = x^{\frac{1}{4}-\frac{1}{5}}\left[\frac{1+\left(\frac{y}{x}\right)^{1/4}}{1+\left(\frac{y}{x}\right)^{1/5}}\right]$$

$$= x^{1/20}\left[\frac{1+\left(\frac{y}{x}\right)^{1/4}}{1+\left(\frac{y}{x}\right)^{1/5}}\right] = x^{1/20} f\left(\frac{y}{x}\right).$$

$\therefore$ $u$ is a homogeneous function in $x$ and $y$ of order $\frac{1}{20}$

$\therefore$ By Euler's theorem, $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = nu$

or $$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \frac{1}{20}\cdot u \qquad \text{...(2)}$$

From (1) and (2), Euler's theorem is verified.

(*ii*) Let $u = ax^2 + 2hxy + by^2$, which is a homogeneous function of degree 2 in $x$ and $y$.

$\therefore$ By Euler's theorem

$$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=2u \qquad ...(1)$$

Now,

$$\frac{\partial u}{\partial x}=2ax+2hy$$

$$\frac{\partial u}{\partial y}=2hx+2by$$

and

$$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=2ax^2+2hxy+2hxy+2by^2$$

$\therefore$

$$=2(ax^2+2hxy+by^2)=2u. \qquad ...(2)$$

From (1) and (2), Euler's theorem is verified.

**Example 2.** *If $u=\dfrac{xy}{x+y}$, show that $x^2\dfrac{\partial^2 u}{\partial x^2}+2xy\dfrac{\partial^2 u}{\partial x\partial y}+y^2\dfrac{\partial^2 u}{\partial y^2}=0.$*

**Solution.** Here $u=\dfrac{xy}{x+y}=\dfrac{x^2\frac{y}{x}}{x\left(1+\frac{y}{x}\right)}=xf\left(\dfrac{y}{x}\right)$

$\therefore$ $u$ is a homogeneous function of degree 1 in $x$ and $y$.

$\therefore$ By Euler's theorem, $x\dfrac{\partial u}{\partial x}+y\dfrac{\partial u}{\partial y}=nu$, where $n=1$

or

$$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=u \qquad ...(1)$$

Differentiating partially w.r.t. $x$, we get

$$x\frac{\partial^2 u}{\partial x^2}+\frac{\partial u}{\partial x}+y\frac{\partial^2 u}{\partial x\partial y}=\frac{\partial u}{\partial x}$$

$\therefore$

$$x\frac{\partial^2 u}{\partial x^2}+y\frac{\partial^2 u}{\partial x\partial y}=0$$

Multiplying by $x$, we get $\quad x^2\dfrac{\partial^2 u}{\partial x^2}+xy\dfrac{\partial^2 u}{\partial x\partial y}=0 \qquad ...(2)$

Interchanging $x$ by $y$, we get $\quad y^2\dfrac{\partial^2 u}{\partial y^2}+xy\dfrac{\partial^2 u}{\partial x\partial y}=0 \qquad ...(3)$

Adding (2) and (3), we get

$$x^2\frac{\partial^2 u}{\partial x^2}+2xy\frac{\partial^2 u}{\partial x\partial y}+y^2\frac{\partial^2 u}{\partial y^2}=0$$

**Example 3.** *If* $u = \sin^{-1}\dfrac{x^2+y^2}{x+y}$, *show that* $x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y} = \tan u$. [K.U. 2016]

**Solution.** Here $$u = \sin^{-1}\frac{x^2+y^2}{x+y}$$

$\Rightarrow$ $$\sin u = \frac{x^2+y^2}{x+y}$$

or $$\sin u = \frac{x^2\left[1+\left(\frac{y}{x}\right)^2\right]}{x\left[1+\frac{y}{x}\right]} = x f\left(\frac{y}{x}\right)$$

$\therefore$ $\sin u$ is a homogeneous function of degree 1 in $x$ and $y$. Thus by Euler's theorem, we have

$$x\frac{\partial}{\partial x}(\sin u) + y\frac{\partial}{\partial y}(\sin u) = 1.\sin u$$

or $$x\cos u\cdot\frac{\partial u}{\partial x} + y\cos u\cdot\frac{\partial u}{\partial y} = \sin u$$

or $$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \frac{\sin u}{\cos u} = \tan u$$

or $$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \tan u.$$

**Example 4.** *If* $u = \log\dfrac{x^2+y^2}{x+y}$, *show that* $x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y} = 1$.

**Solution.** Here $$u = \log\frac{x^2+y^2}{x+y}$$ [*Non-homogeneous function*]

$\therefore$ $$e^u = \frac{x^2+y^2}{x+y}$$

Let $$z = e^u$$

$\therefore$ $$z = \frac{x^2\left(1+\frac{y^2}{x^2}\right)}{x\left(1+\frac{y}{x}\right)} = x f\left(\frac{y}{x}\right)$$

Here $z$ is a homogeneous function of degree 1 in $x$ and $y$. Thus by Euler's theorem, we have

$$x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = 1\,.\,z$$



$$x\frac{\partial}{\partial x}(e^u) + y\frac{\partial}{\partial y}(e^u) = e^u$$ [$\because z = e^u$]

or $$xe^u\frac{\partial u}{\partial x} + ye^u\frac{\partial u}{\partial y} = e^u$$

or Dividing both sides by $e^u$, we get $x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y} = 1$.

**Example 5.** *If* $u = x\phi\left(\dfrac{y}{x}\right) + \psi\left(\dfrac{y}{x}\right)$, *show that*

$$x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x\partial y} + y^2\frac{\partial^2 u}{\partial y^2} = 0.$$ [M.D.U. 2015]

**Solution.** Let $z_1 = x\phi\left(\dfrac{y}{x}\right)$, which is a homogeneous function of degree 1 in $x$ and $y$.

$\therefore$ $$x^2\frac{\partial^2 z_1}{\partial x^2} + 2xy\frac{\partial^2 z_1}{\partial x\partial y} + y^2\frac{\partial^2 z_1}{\partial y^2} = n(n-1)z_1 = 1\cdot(1-1)\cdot z_1 = 0 \quad ...(1)$$

Let $z_2 = \psi\left(\dfrac{y}{x}\right) = x^0\,\psi\left(\dfrac{y}{x}\right)$, which is a homogeneous function of degree 0 in $x$ and $y$.

$\therefore$ $$x^2\frac{\partial^2 z_2}{\partial x^2} + 2xy\frac{\partial^2 z_2}{\partial x\partial y} + y^2\frac{\partial^2 z_2}{\partial y^2} = n(n-1)z_2 = 0(0-1)z_2 = 0 \quad ...(2)$$

Adding (1) and (2), we get

$$x^2\frac{\partial^2}{\partial x^2}(z_1+z_2) + 2xy\frac{\partial^2}{\partial x\partial y}(z_1+z_2) + y^2\frac{\partial^2}{\partial y^2}(z_1+z_2) = 0$$

or $$x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x\partial y} + y^2\frac{\partial^2 u}{\partial y^2} = 0.$$ [$\because z_1 + z_2 = u$]

**Example 6.** *If* $u = \tan^{-1}\dfrac{x^3+y^3}{x-y}$, *prove that*

(i) $x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y} = \sin 2u$ [K.U. 2014; M.D.U. 2010]

(ii) $x^2\dfrac{\partial^2 u}{\partial x^2} + 2xy\dfrac{\partial^2 u}{\partial x\partial y} + y^2\dfrac{\partial^2 u}{\partial y^2} = \sin 2u\,(1-4\sin^2 u)$ [K.U. 2011, 06, 05]

**Solution.** (i) Here $$u = \tan^{-1}\frac{x^3+y^3}{x-y}$$

$\Rightarrow$
$$\tan u = \frac{x^3 + y^3}{x - y} = \frac{x^3\left[1+\left(\frac{y}{x}\right)^3\right]}{x\left[1-\left(\frac{y}{x}\right)\right]} = x^2 f\left(\frac{y}{x}\right)$$

$\therefore$ **tan $u$ is a homogeneous function of degree 2 in $x$ and $y$.**

$\therefore$ **By Euler's theorem, we have**

$$x\frac{\partial}{\partial x}(\tan u) + y\frac{\partial}{\partial y}(\tan u) = 2\tan u$$

or
$$x\sec^2 u\cdot\frac{\partial u}{\partial x} + y\sec^2 u\cdot\frac{\partial u}{\partial y} = 2.\tan u$$

or
$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \frac{2\tan u}{\sec^2 u}$$

*i.e.,*
$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \sin 2u$$

(*ii*) Differentiating (1) partially w.r.t. $x$, we get

$$x\frac{\partial^2 u}{\partial x^2} + \frac{\partial u}{\partial x} + y\frac{\partial^2 u}{\partial x\partial y} = 2\cos 2u\frac{\partial u}{\partial x}$$

Multiplying by $x$, we have

$$x^2\frac{\partial^2 u}{\partial x^2} + x\frac{\partial u}{\partial x} + xy\frac{\partial^2 u}{\partial x\partial y} = \left(2x\frac{\partial u}{\partial x}\right)\cos 2u$$

Differentiating (1) partially w.r.t. $y$, we get

$$x\frac{\partial^2 u}{\partial x\partial y} + \frac{\partial u}{\partial y} + y\frac{\partial^2 u}{\partial y^2} = 2\cos 2u\frac{\partial u}{\partial y}$$

Multiplying by $y$, we have

$$xy\frac{\partial^2 u}{\partial x\partial y} + y\frac{\partial u}{\partial y} + y^2\frac{\partial^2 u}{\partial y^2} = \left(2y\frac{\partial u}{\partial y}\right)\cos 2u$$

Adding (2) and (3), we have

$$x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x\partial y} + y^2\frac{\partial^2 u}{\partial y^2} + \left(x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y}\right) = 2\left(x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y}\right)\cdot\cos 2u$$

Using (1), we get

$$x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x\partial y} + y^2\frac{\partial^2 u}{\partial y^2} + \sin 2u = 2\,.\,\sin 2u\,.\,\cos 2u$$

$$x^2 \frac{\partial^2 u}{\partial x^2} + 2xy \frac{\partial^2 u}{\partial x \partial y} + y^2 \frac{\partial^2 u}{\partial y^2} = 2 \sin 2u \cos 2u - \sin 2u$$

or

$$= \sin 2u \, [2 \cos 2u - 1]$$

$$= \sin 2u \, [2 (1 - 2 \sin^2 u) - 1]$$

$$= \sin 2u \, [2 - 4 \sin^2 u - 1]$$

$$\therefore \quad x^2 \frac{\partial^2 u}{\partial x^2} + 2xy \frac{\partial^2 u}{\partial x \partial y} + y^2 \frac{\partial^2 u}{\partial y^2} = \sin 2u \, [1 - 4 \sin^2 u].$$

**Definition.** *If $u = f(x, y)$ is differentiable, then the differential or* **total differential** *of u is defined by*

$$du = \frac{\partial u}{\partial x} dx + \frac{\partial u}{\partial y} dy \quad ...(8)$$

where $dx$ and $dy$ are independent variables.

Equation (8) can be written as $du = \left(\frac{\partial}{\partial x} dx + \frac{\partial}{\partial y} dy\right) u.$

Thus we see that the differential $du$ is a function of four independent variables, namely $(x, y)$ and $dx, dy$, where $(x, y)$ are the co-ordinate of the point at which the partial derivatives are to be calculated and $dx$ and $dy$ are differentials of the independent variables. If we identify differentials with actual increments, then

$$dx = \delta x,\ dy = \delta y \quad ...(9)$$

From (8) and (9), we get $\quad du = \frac{\partial u}{\partial x} \delta x + \frac{\partial u}{\partial y} \delta y \quad ...(10)$

Substituting (10) in (7), we get

$$\delta u = \delta f = du + \varepsilon_1 \delta x + \varepsilon_2 \delta y \quad ...(11)$$

**Remark.**

It may be observed that the total differential of $u = f(x, y)$ *i.e.*, $du$ is the sum of the two partial differentials $\frac{\partial u}{\partial x} dx$ and $\frac{\partial u}{\partial y} dy$.

Similarly if $u = f(x, y, z)$, then $du = \frac{\partial u}{\partial x} dx + \frac{\partial u}{\partial y} dy + \frac{\partial u}{\partial z} dz$ and so on.

It is assumed that the variables in the function $u = f(x, y)$ are independent, so that their respective increments are constants in their own right *i.e.*, $dx$ and $dy$ are constant.

## 5.9. SUCCESSIVE TOTAL DIFFERENTIALS

Let $u = f(x, y)$ be a function of two independent variables $x$ and $y$ having partial derivatives continuous in its domain $A \subset R^2$. Then the first total differential of $u$ is

$$du = \frac{\partial u}{\partial x} dx + \frac{\partial u}{\partial y} dy \quad ...(1)$$

The second total differential *i.e.*, the differential of $du$ is denoted by $d(du) = d^2u$. Thus

$$d^2u = d(du) = d\left[\frac{\partial u}{\partial x} dx + \frac{\partial u}{\partial y} dy\right]$$



$$= d\left(\frac{\partial u}{\partial x}\right) dx + d\left(\frac{\partial u}{\partial y}\right) dy \qquad [\because dx \text{ and } dy \text{ are constants}]$$

$$= \left(\frac{\partial^2 u}{\partial x^2} dx + \frac{\partial^2 u}{\partial y \partial x} dy\right) dx + \left(\frac{\partial^2 u}{\partial x \partial y} dx + \frac{\partial^2 u}{\partial y^2} dy\right) dy$$

$$= \frac{\partial^2 u}{\partial x^2} (dx)^2 + 2\frac{\partial^2 u}{\partial x \partial y} dx dy + \frac{\partial^2 u}{\partial y^2} (dy)^2 \qquad \left[\text{We assume that } \frac{\partial^2 u}{\partial x \partial y} = \frac{\partial^2 u}{\partial y \partial x}\right]$$

which may be written in symbolic form as

$$d^2u = \left(\frac{\partial}{\partial x} dx + \frac{\partial}{\partial y} dy\right)^2 u$$

Proceeding similarly, we can calculate the total differentials of higher order and in general, we have

$$d^n u = \left(\frac{\partial}{\partial x} dx + \frac{\partial}{\partial y} dy\right)^n u.$$

## 5.10. COMPOSITE FUNCTIONS

*If u is a function of the variables x, y and these variables x, y are themselves functions of another variable t, then u is said to be a* **composite function** *of the variable t.* In other words, the relation $u = f(x, y)$; $x = \phi(t)$, $y = \psi(t)$ define $u$ as a composite function of $t$.

Also the relations $u = f(x, y)$; $x = \phi(r, s)$, $y = \psi(r, s)$ define $u$ as a composite function of $r$ and $s$.

### 5.10.1. Theorem.

**If $u$ is a composite function of $t$ defined by the relations $u = f(x, y)$; $x = \phi(t), y = \psi(t)$ such that $u$ possesses continuous first order partial derivatives w.r.t. $x$ and $y$; and $x, y$ possesses continuous derivatives w.r.t. '$t$', then**

$$\frac{du}{dt} = \frac{\partial u}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial u}{\partial y} \cdot \frac{dy}{dt}. \quad ...(1)$$

**Proof.** Here $u = f(x, y)$; $x = \phi(t), y = \psi(t)$

Giving increment $\delta t$ to $t$, let the corresponding increments in $x, y$ and $u$ be $\delta x, \delta y$ and $\delta u$ respectively.

$\therefore \quad u + \delta u = f(x + \delta y, y + \delta y)$

$\Rightarrow \quad \delta u = f(x + \delta x, y + \delta y) - f(x, y)$

## 5.11. CHANGE OF VARIABLES

Let $u = f(x, y)$ such that $x = \phi(r, s)$ and $y = \psi(r, s)$.

It is required to change expressions involving $u, x, y$ $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}$ etc. to expressions involving $u, r, s, \frac{\partial u}{\partial r}, \frac{\partial u}{\partial s}$ etc.

If $s$ is regarded as a constant, then $x, y, u$ will be functions of $r$ alone.

$$\therefore \quad \frac{\partial u}{\partial r} = \frac{\partial u}{\partial x}\cdot\frac{\partial x}{\partial r} + \frac{\partial u}{\partial y}\cdot\frac{\partial y}{\partial r} \quad ...(1)$$

Here the ordinary derivatives have been replaced by partial derivative as $u, x, y$, are functions of $r$ and $s$.

Similarly if $r$ is constant, then

$$\frac{\partial u}{\partial s} = \frac{\partial u}{\partial x}\cdot\frac{\partial x}{\partial s} + \frac{\partial u}{\partial y}\cdot\frac{\partial y}{\partial s} \quad ...(2)$$

On solving (1) and (2) as simultaneous equations in $\frac{\partial u}{\partial x}$ and $\frac{\partial u}{\partial y}$, we get their values in terms of $\frac{\partial u}{\partial r}, \frac{\partial u}{\partial s}, u, r, s$.

## SOLVED EXAMPLES

**Example 1.** ***Find the total derivative of u with respect to t, when $u = e^x \sin y$, where $x = \log t$, $y = t^2$. Also verify by direct calculations.*** [M.D.U. 2011]

**Solution.** Here $\quad u = e^x \sin y \quad ...(1)$

$$\left.\begin{aligned} x &= \log t \\ y &= t^2 \end{aligned}\right\} \quad ...(2)$$

Now, $u$ is a function of $x, y$ and $x, y$ are themselves function of $t$. Therefore, $u$ is a composite function of $t$.

$$\therefore \quad \frac{du}{dt} = \frac{\partial u}{\partial x}\cdot\frac{dx}{dt} + \frac{\partial u}{\partial y}\cdot\frac{dy}{dt}$$

$$= e^x \sin y.\ \frac{1}{t} + e^x \cos y \cdot 2t$$

$$= t\cdot\sin t^2.\ \frac{1}{t} + t\cdot\cos t^2\cdot 2t \quad [\because x = \log t \Rightarrow e^x = t]$$

$$= \sin t^2 + 2t^2 \cos t^2 \quad ...(3)$$

**By direct method :**

$$u = e^x \sin y = e^{\log t} \sin t^2 = t \sin t^2$$

Differentiating w.r.t. $t$, $\dfrac{du}{dt} = \sin t^2 + 2t^2 \cos t^2$

Hence the verification.

**Example 2.** *If* $V = f(y-z, z-x, x-y)$, *prove that* $\dfrac{\partial V}{\partial x} + \dfrac{\partial V}{\partial y} + \dfrac{\partial V}{\partial z} = 0$. [K.U. 2016]

**Solution.** Let $r = y - z,\ \ s = z - x,\ \ t = x - y$ so that $V = f(r, s, t)$, where $r, s, t$ are functions of $x, y, z$.

$\therefore$ V is a composite function of $x, y, z$.

$$\therefore \quad \frac{\partial V}{\partial x} = \frac{\partial V}{\partial r}\cdot\frac{\partial r}{\partial x} + \frac{\partial V}{\partial s}\cdot\frac{\partial s}{\partial x} + \frac{\partial V}{\partial t}\cdot\frac{\partial t}{\partial x} \quad ...(1)$$

But $r = y - z, s = z - x, t = x - y$

$$\therefore \quad \frac{\partial r}{\partial x} = 0,\ \frac{\partial s}{\partial x} = -1,\ \frac{\partial t}{\partial x} = 1$$

Putting these values in (1), we get

$$\frac{\partial V}{\partial x} = -\frac{\partial V}{\partial s} + \frac{\partial V}{\partial t} \quad ...(2)$$

Similarly,
$$\frac{\partial V}{\partial y} = -\frac{\partial V}{\partial t} + \frac{\partial V}{\partial r} \quad ...(3)$$

and
$$\frac{\partial V}{\partial z} = -\frac{\partial V}{\partial r} + \frac{\partial V}{\partial s} \quad ...(4)$$

Adding (2), (3) and (4), we get

$$\frac{\partial V}{\partial x} + \frac{\partial V}{\partial y} + \frac{\partial V}{\partial z} = 0.$$

**Example 3.** *If* $z = f(x, y)$ *and* $x = e^u + e^{-v}, y = e^{-u} - e^v$; *prove that*

$$\frac{\partial z}{\partial u} - \frac{\partial z}{\partial v} = x\frac{\partial z}{\partial x} - y\frac{\partial z}{\partial y} \quad \text{[K.U. 2007]}$$

**Solution.** Here $z$ is a function of $x, y$ and $x, y$ are themselves the functions of $u, v$. Therefore $z$ is a composite function of $u$ and $v$

$$\therefore \quad \frac{\partial z}{\partial u} = \frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial u} + \frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial u} \quad ...(1)$$

$$\frac{\partial z}{\partial v} = \frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial v} + \frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial v} \quad ...(2)$$



Now, $\dfrac{\partial x}{\partial u} = e^u,\ \dfrac{\partial x}{\partial v} = -e^{-v},\ \dfrac{\partial y}{\partial u} = -e^{-u}$ and $\dfrac{\partial y}{\partial v} = -e^v$

Putting these values in (1) and (2), we have

$$\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x}\cdot e^u + \frac{\partial z}{\partial y}(-e^{-u}) \quad ...(3)$$

$$\frac{\partial z}{\partial v} = \frac{\partial z}{\partial x}(-e^{-v}) + \frac{\partial z}{\partial y}(-e^v) \quad ...(4)$$

Subtracting (4) from (3), we get

$$\frac{\partial z}{\partial u} - \frac{\partial z}{\partial v} = (e^u + e^{-v})\frac{\partial z}{\partial x} - (e^{-u} - e^v)\frac{\partial z}{\partial y}$$

or
$$\frac{\partial z}{\partial u} - \frac{\partial z}{\partial v} = x\cdot\frac{\partial z}{\partial x} - y\frac{\partial z}{\partial y}.$$

**Example 4.** *If* $x = u + v,\ y = uv$ *and* $z$ *is a function of* $x$ *and* $y$, *show that*

$$u\frac{\partial z}{\partial u} + v\frac{\partial z}{\partial v} = x\frac{\partial z}{\partial x} + 2y\frac{\partial z}{\partial y}.$$

**Solution.** Here $z$ is a function of $x$, $y$ and $x, y$ are themselves the functions of $u, v$. Therefore $z$ is a composite function of $u, v$

$$\therefore \quad \frac{\partial z}{\partial u} = \frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial u} + \frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial u} \quad ...(1)$$

and
$$\frac{\partial z}{\partial v} = \frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial v} + \frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial v} \quad ...(2)$$

Now,
$$\frac{\partial x}{\partial u} = 1,\ \frac{\partial x}{\partial v} = 1,\ \frac{\partial y}{\partial u} = v,\ \frac{\partial y}{\partial v} = u.$$

Putting these values in (1) and (2), we have

$$\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x}(1) + \frac{\partial z}{\partial y}\cdot v \quad ...(3)$$

and
$$\frac{\partial z}{\partial v} = \frac{\partial z}{\partial x}(1) + \frac{\partial z}{\partial y}\cdot u \quad ...(4)$$

Multiplying (3) by $u$ and (4) by $v$ and adding, we get

$$u\frac{\partial z}{\partial u} + v\frac{\partial z}{\partial v} = (u+v)\frac{\partial z}{\partial x} + (2uv)\frac{\partial z}{\partial y}$$

$$\therefore \quad u\frac{\partial z}{\partial u} + v\cdot\frac{\partial z}{\partial v} = x\frac{\partial z}{\partial x} + 2y\frac{\partial z}{\partial y}.$$

**Example 5.** *If $z = 2u^2 - v^2 + 3w^2$, where $u = xe^y$, $v = ye^{-x}$, $w = \frac{y}{x}$, find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$.* [M.D.U. 2007]

**Solution.** Here $z = 2u^2 - v^2 + 3w^2$ i.e., $z$ is a function of $u$, $v$ and $w$

and $u = xe^y,\ v = ye^{-x},\ w = \frac{y}{x}.$

$$\therefore \quad \frac{\partial z}{\partial x} = \frac{\partial z}{\partial u}\cdot\frac{\partial u}{\partial x} + \frac{\partial z}{\partial v}\cdot\frac{\partial v}{\partial x} + \frac{\partial z}{\partial w}\cdot\frac{\partial w}{\partial x} \quad ...(1)$$

$$\text{and} \quad \frac{\partial z}{\partial y} = \frac{\partial z}{\partial u}\cdot\frac{\partial u}{\partial y} + \frac{\partial z}{\partial v}\cdot\frac{\partial v}{\partial y} + \frac{\partial z}{\partial w}\cdot\frac{\partial w}{\partial y} \quad ...(2)$$

Now, $z = 2u^2 - v^2 + 3w^2$

$$\therefore \quad \frac{\partial z}{\partial u} = 4u;\ \frac{\partial z}{\partial v} = -2v \text{ and } \frac{\partial z}{\partial w} = 6w \quad ...(3)$$

Again, $u = xe^y$

$$\therefore \quad \frac{\partial u}{\partial x} = e^y \text{ and } \frac{\partial u}{\partial y} = xe^y \quad ...(4)$$

Also, $v = ye^{-x}$

$$\therefore \quad \frac{\partial v}{\partial x} = -ye^{-x} \text{ and } \frac{\partial v}{\partial y} = e^{-x} \quad ...(5)$$

Also, $w = \frac{y}{x}$

$$\therefore \quad \frac{\partial w}{\partial x} = -\frac{y}{x^2} \text{ and } \frac{\partial w}{\partial y} = \frac{1}{x} \quad ...(6)$$

Substituting the values from eqns. (3), (4), (5) and (6) in eqn. (1), we get

$$\frac{\partial z}{\partial x} = 4u.e^y + (-2v)(-ye^{-x}) + 6w\left(-\frac{y}{x^2}\right)$$

Putting the values of $u$, $v$ and $w$, we have

$$\frac{\partial z}{\partial x} = 4(xe^y).e^y - 2ye^{-x}(-ye^{-x}) + 6\frac{y}{x}\left(-\frac{y}{x^2}\right)$$

$$\frac{\partial z}{\partial x} = 4xe^{2y} + 2y^2e^{-2x} - 6\frac{y^2}{x^3}.$$



Substituting the values from eqns. (3), (4), (5) and (6) in eqn. (2), we get

$$\frac{\partial z}{\partial y} = 4u.(xe^y) + (-2v)(e^{-x}) + 6w.\frac{1}{x}$$

Putting the values of $u$, $v$ and $w$, we have

$$\frac{\partial z}{\partial y} = 4xe^y(xe^y) - 2ye^{-x}(e^{-x}) + 6\frac{y}{x}.\frac{1}{x}$$

$$\text{or} \quad \frac{\partial z}{\partial y} = 4x^2e^{2y} - 2ye^{-2x} + 6\frac{y}{x^2}.$$

**Example 6.** *If $z = \tan^{-1}\left(\frac{x}{y}\right)$, $x = u + v$, $y = u - v$; show that $\frac{\partial z}{\partial u} + \frac{\partial z}{\partial v} = \frac{u - v}{u^2 + v^2}$.*

**Solution.** Here $z$ is a function of $x$, $y$ and $x$, $y$ are themselves functions of $u$, $v$. Therefore $z$ is a composite function of $u$, $v$

$$\therefore \quad \frac{\partial z}{\partial u} = \frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial u} + \frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial u} \quad ...(1)$$

$$\text{and} \quad \frac{\partial z}{\partial v} = \frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial v} + \frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial v} \quad ...(2)$$

Now, $x = u + v$, $y = u - v \Rightarrow \frac{\partial x}{\partial u} = 1,\ \frac{\partial x}{\partial v} = 1,\ \frac{\partial y}{\partial u} = 1,\ \frac{\partial y}{\partial v} = -1.$

Since $z = \tan^{-1}\left(\frac{x}{y}\right)$

$$\therefore \quad \frac{\partial z}{\partial x} = \frac{1}{1 + \left(\frac{x}{y}\right)^2}\cdot\left(\frac{1}{y}\right) = \frac{y}{y^2 + x^2} = \frac{u - v}{(u - v)^2 + (u + v)^2}$$

$$= \frac{u - v}{2(u^2 + v^2)}$$

$$\text{and} \quad \frac{\partial z}{\partial y} = \frac{1}{1 + \left(\frac{x}{y}\right)^2}\left(\frac{-x}{y^2}\right) = \frac{-x}{y^2 + x^2} = \frac{-(u + v)}{(u - v)^2 + (u + v)^2}$$

$$= \frac{-(u + v)}{2(u^2 + v^2)}$$

Putting the above values in (1) and (2), we have

$$\frac{\partial z}{\partial u} = \frac{u - v}{2(u^2 + v^2)}.1 - \frac{u + v}{2(u^2 + v^2)}.1 = \frac{-2v}{2(u^2 + v^2)} \quad ...(3)$$

and $$\frac{\partial z}{\partial v} = \frac{u-v}{2(u^2+v^2)} \cdot 1 - \frac{u+v}{2(u^2+v^2)} \cdot (-1) = \frac{2u}{2(u^2+v^2)}$$

Adding (3) and (4), we get

$$\frac{\partial z}{\partial u} + \frac{\partial z}{\partial v} = \frac{-2v}{2(u^2+v^2)} + \frac{2u}{2(u^2+v^2)}$$

$$= \frac{2u-2v}{2(u^2+v^2)} = \frac{u-v}{u^2+v^2}.$$

**Example 7.** *If $w = f(x, y)$, $x = r\cos\theta$, $y = r\sin\theta$; show that*

$$\left(\frac{\partial w}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial w}{\partial \theta}\right)^2 = \left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2.$$ [K.U. 201...]

**Solution.** Here $w$ is a function of $x, y$ and $x, y$ are themselves functions of $r, \theta$. Therefore $w$ is a composite function of $r, \theta$

$$\therefore \quad \frac{\partial w}{\partial r} = \frac{\partial w}{\partial x}\cdot\frac{\partial x}{\partial r} + \frac{\partial w}{\partial y}\cdot\frac{\partial y}{\partial r} = \frac{\partial f}{\partial x}\cdot\frac{\partial x}{\partial r} + \frac{\partial f}{\partial y}\cdot\frac{\partial y}{\partial r} \quad ...(1)$$

and $$\frac{\partial w}{\partial \theta} = \frac{\partial w}{\partial x}\cdot\frac{\partial x}{\partial \theta} + \frac{\partial w}{\partial y}\cdot\frac{\partial y}{\partial \theta} = \frac{\partial f}{\partial x}\cdot\frac{\partial x}{\partial \theta} + \frac{\partial f}{\partial y}\cdot\frac{\partial y}{\partial \theta} \quad ...(2)$$

Now, $$\frac{\partial x}{\partial r} = \cos\theta, \qquad \frac{\partial x}{\partial \theta} = -r\sin\theta,$$

$$\frac{\partial y}{\partial r} = \sin\theta \quad \text{and} \quad \frac{\partial y}{\partial \theta} = r\cos\theta$$

Putting these values in (1) and (2) we have

$$\frac{\partial w}{\partial r} = \frac{\partial f}{\partial x}\cdot\cos\theta + \frac{\partial f}{\partial y}\cdot\sin\theta \quad ...(3)$$

and $$\frac{\partial w}{\partial \theta} = \frac{\partial f}{\partial x}\cdot(-r\sin\theta) + \frac{\partial f}{\partial y}\cdot(r\cos\theta)$$

or $$\frac{1}{r}\frac{\partial w}{\partial \theta} = -\frac{\partial f}{\partial x}\sin\theta + \frac{\partial f}{\partial y}\cdot\cos\theta \quad ...(4)$$

Squaring and adding (3) and (4), we get

$$\left(\frac{\partial w}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial w}{\partial \theta}\right)^2 = \left(\frac{\partial f}{\partial x}\right)^2(\cos^2\theta + \sin^2\theta) + \left(\frac{\partial f}{\partial y}\right)^2(\sin^2\theta + \cos^2\theta)$$

$$\therefore \quad \left(\frac{\partial w}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial w}{\partial \theta}\right)^2 = \left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2.$$